

# THE DECOMPOSITION OF THE HYPERMETRIC CONE INTO $L$ -DOMAINS

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**ABSTRACT.** The hypermetric cone  $\text{HYP}_{n+1}$  is the parameter space of basic Delaunay polytopes of  $n$ -dimensional lattice. If one fixes one Delaunay polytope of the lattice then there are only a finite number of possibilities for the full Delaunay tessellations. So, the cone  $\text{HYP}_{n+1}$  is the union of a finite set of  $L$ -domains, i.e. of parameter space of full Delaunay tessellations.

In this paper, we study this partition of the hypermetric cone into  $L$ -domains. In particular, we prove that the cone  $\text{HYP}_{n+1}$  of hypermetrics on  $n+1$  points contains exactly  $\frac{1}{2}n!$  principal  $L$ -domains. We give a detailed description of the decomposition of  $\text{HYP}_{n+1}$  for  $n = 2, 3, 4$  and a computer result for  $n = 5$  (see Table 2). Remarkable properties of the root system  $D_4$  are key for the decomposition of  $\text{HYP}_5$ .

## 1. INTRODUCTION

An  $n$ -dimensional lattice  $L$  is a subgroup of  $\mathbb{R}^n$  of the form  $L = v_1\mathbb{Z} + \dots + v_n\mathbb{Z}$  with  $(v_1, \dots, v_n)$  a basis of  $\mathbb{R}^n$ . Let  $S(c, r)$  be a sphere in  $\mathbb{R}^n$  with center  $c$  and radius  $r$ . Then,  $S(c, r)$  is said to be a *Delaunay sphere* in the lattice  $L$  if the following two conditions hold:

- (i)  $\|v - c\| \geq r$  for all  $v \in L$ ,
- (ii) the set  $S(c, r) \cap L$  has affine rank  $n + 1$ .

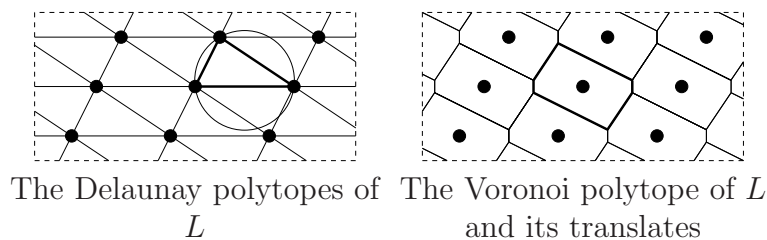
The  $n$ -dimensional polytope  $P$ , which is defined as the convex hull of the set  $S(c, r) \cap L$ , is called a *Delaunay polytope* of rank  $n$ . The Delaunay polytopes of rank  $n$  form a face-to-face tiling of  $\mathbb{R}^n$ . The *Voronoi polytope*  $P_V(L)$  of a lattice  $L$  is the set of points, whose closest element in  $L$  is 0. Its vertices are centers of Delaunay polytopes of  $L$ . The polytope  $P_V(L)$  forms a tiling of  $\mathbb{R}^n$  under translation by  $L$ , i.e. it is a *parallelohedron* (see Figure 1).

The cones  $\mathcal{S}_{>0}^n$ ,  $\mathcal{S}_{\geq 0}^n$  are respectively the cone of positive definite, positive semi-definite  $n \times n$  matrices. The *rational closure*  $\mathcal{S}_{\text{rat} \geq 0}^n$  of  $\mathcal{S}_{>0}^n$  is defined as the positive semidefinite matrices, whose kernel is defined by rational equalities (see [DSV06]). Given a basis  $B = (v_i)_{1 \leq i \leq n}$  of a lattice  $L$ , we associate the Gram

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FIGURE 1. A lattice  $L \subset \mathbb{R}^2$  and the induced partitions

matrix  $a = B^T B \in \mathcal{S}_{>0}^n$ . On the other hand if  $a \in \mathcal{S}_{>0}^n$ , then we can find an invertible real matrix  $B$  such that  $a = B^T B$ , i.e.  $B$  is the basis of a lattice  $L = B\mathbb{Z}^n$  with Gram matrix  $a$ . So, we can replace the study of Delaunay polytopes of  $L$  for the standard scalar product by the study of Delaunay polytopes of  $\mathbb{Z}^n$  for the scalar product  $x^T a x$ . If one takes another basis  $B'$  of  $L$ , then  $B' = BP$  for some  $P \in \text{GL}_n(\mathbb{Z})$  and one has  $a' = P^T a P$ , i.e.  $a$  and  $a'$  are *arithmetically equivalent*. In other words, the study of  $n$ -dimensional lattices up to isometric equivalence is the same as the study of positive definite  $n \times n$  symmetric matrices, up to arithmetic equivalence. In [DSV06] it is proved that if  $a \in \mathcal{S}_{\geq 0}^n$ , then one can define, possibly infinite, Delaunay polytopes of  $\mathbb{Z}^n$  for  $x^T a x$  if and only if  $a \in \mathcal{S}_{\text{rat} \geq 0}^n$ .

Given a polytope  $P$  of  $\mathbb{Z}^n$  the condition that it is a Delaunay polytope for the norm  $x^T a x$  translates to linear equalities and strict inequalities on the coefficients of  $a$ . An  $L$ -domain is the convex cone of all matrices  $a \in \mathcal{S}_{>0}^n$  such that  $\mathbb{Z}^n$  has the same Delaunay tessellation for  $x^T a x$  (Details see, for example, in [Vo08, DL97, DSV06]). Voronoi proved that the cone  $\mathcal{S}_{>0}^n$  is partitioned into polyhedral  $L$ -domains. An  $L$ -domain of maximal dimension  $\frac{1}{2}n(n+1)$  is called *primitive*. An  $L$ -domain is primitive if and only if the Delaunay tiling related to it consists only of simplices. Each non-primitive  $L$ -domain is an open face of the closure of a primitive one. In particular, an extreme ray of the closure of an  $L$ -domain is a non-primitive one-dimensional  $L$ -domain. The group  $\text{GL}_n(\mathbb{Z})$  acts on the  $L$ -domains of  $\mathcal{S}_{>0}^n$  by  $\mathcal{D} \mapsto P^T \mathcal{D} P$ , and there is a finite number of orbits of  $L$ -domains, called  *$L$ -types*. The geometric viewpoint is most useful for thinking, and drawings about lattice and the Gram matrix viewpoint is the most suitable to machine computations.

A *metric* on the set  $\{0, 1, \dots, n\}$  is a function  $d$  such that  $d(x, x) = 0$ ,  $d(x, y) = d(y, x)$  and  $d(x, y) \leq d(x, z) + d(z, y)$ . A metric  $d$  is an *hypermetric* if it satisfies the inequalities

$$(1) \quad H_z(d) = \sum_{0 \leq i < j \leq n} z_i z_j d(i, j) \leq 0$$

for all integral vectors  $z \in \mathbb{Z}^{n+1}$  such that  $\sum_{i=0}^n z_i = 1$ . The set of all hypermetrics on  $n$  points  $\{0, \dots, n-1\}$  is denoted by  $\text{HYP}_n$ .

$n$	representative of orbits of facets of $\text{HYP}_n$
3	$(1, 1, -1)$ (triangle inequality)
4	$(1, 1, -1, 0)$
5	$(1, 1, -1, 0, 0)$ and $(1, 1, 1, -1, -1)$ (pentagonal inequality)
6	$(1, 1, -1, 0, 0, 0)$ , $(1, 1, 1, -1, -1, 0)$ , $(1, 1, 1, 1, -1, -2)$ and $(2, 1, 1, -1, -1, -1)$

TABLE 1. The facets of  $\text{HYP}_n$  for  $n \leq 6$ 

The group  $\text{Sym}(n)$  acts on  $\text{HYP}_n$ ; it is proved in [DGP06] that there is no other symmetries if  $n \neq 4$ . It is proved in [DL97] that  $\text{HYP}_{n+1}$  is polyhedral, i.e. among the infinite set of inequalities of the form (1), a finite number suffices to get all facets. This result can be proved in many different ways, see [DL97, Theorem 14.2.1]; the second proof uses that the image  $\xi(\text{HYP}_{n+1})$  is the union of a finite number of  $L$ -domains. The purpose of this article is to investigate such decompositions of  $\text{HYP}_{n+1}$ .

The set of orbits of facets of  $\text{HYP}_n$  for  $n \leq 6$  is given in Table 1,  $\text{HYP}_7$  has 14 orbits of facets (see [DL97, Ba99, RB]) and the list is not known for  $n \geq 8$ . An inequality of (1) is called  $k$ -gonal if  $\sum_{i=0}^n |z_i| = k$ . 3-gonal and 5-gonal inequalities are also called *triangle* and *pentagonal* inequalities, respectively.

A Delaunay polytope  $P$  of a lattice  $L$  is called *generating* if the smallest, for the inclusion relation, lattice containing  $V(P)$  is  $L$ . Moreover, if there exist a family  $(v_0, \dots, v_n)$  of vertices of  $P$  such that for any  $v \in L$  there exist  $\alpha_i \in \mathbb{Z}$  with

$$1 = \sum_{i=0}^n \alpha_i, v = \sum_{i=0}^n \alpha_i v_i$$

then  $P$  is called *basic* and  $(v_0, \dots, v_n)$  is an *affine basis*. Given such an affine basis, we define the distance  $d(i, j) = \|v_i - v_j\|^2$  and we have

$$H_b(d) = \sum_{0 \leq i < j \leq n} b_i b_j d(i, j) = (r^2 - \|\sum_{i=0}^n b_i v_i - c\|^2) \leq 0,$$

where  $r$  and  $c$  relate to the circumscribing sphere  $S(c, r)$  of  $P$ . So the hypermetric inequalities correspond to the inequalities determining a family  $(v_0, \dots, v_n)$  to be an affine basis of a Delaunay polytope. Moreover we have  $H_b(d) = 0$  if and only if  $\sum_{i=0}^n b_i v_i$  is a vertex of  $P$ . In other words the hypermetric cone  $\text{HYP}_{n+1}$  is the parameter space of a basic simplex in  $\mathbb{Z}^n$ . We refer for proofs to [As82, DL97]. In practice, if  $(v_0, \dots, v_n)$  is an affine basis, we can replace it by  $v_0 = 0$ ,  $v_i = e_i$  and call the corresponding simplex *main*. At this point we should note that the hypermetric cone is just one possibility for a parameter space of Delaunay polytopes. Following [Er75, Er92], define  $M_{2,n}$  to be the space of real polynomials

of  $n$  variables with degree at most 2. We then have

$$C_n = \{f \in M_{2,n} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{Z}^n\}$$

If  $P$  is a Delaunay polytope of rank  $k \leq n$ , then we define

$$C_n(P) = \{f \in C_n \mid f(x) = 0 \text{ for all } x \in \text{vert } P\}$$

Those cones were used in [DR03] to find some so-called *perfect* Delaunay polytopes. Note that if  $P = \{0, e_1, \dots, e_n\}$ , then the cone  $C_n(P)$  is isomorphic to the cone  $\text{HYP}_{n+1}$ .

The covariance map  $\xi : d \rightarrow a$  transforms a hypermetric  $d$  on  $n + 1$  points  $i$ ,  $0 \leq i \leq n$ , of a set  $X$  into an  $n \times n$  positive semidefinite symmetric matrix  $a$  as follows:

$$a_{ij} = \xi(d(i, j)) = \frac{1}{2}(d(0, i) + d(0, j) - d(i, j))$$

(see [DL97, Section 5.2]). The covariance  $\xi$  maps the hypermetric cone  $\text{HYP}_{n+1}$  into  $\mathcal{S}_{\text{rat} \geq 0}^n$ . Note that there are  $n + 1$  distinct such maps depending on which point of  $\{0, e_1, \dots, e_n\}$  is chosen as the zero point.

## 2. DECOMPOSITION METHODS

Recall that the Delaunay tiling related to a Gram matrix  $a$  from a primitive  $L$ -domain  $\mathcal{D}$  consists of simplices. The set of Delaunay simplices of the tiling containing the common lattice point  $0$  is the star  $St_0$ . By translations, along  $\mathbb{Z}^n$ , the star  $St_0$  determines fully the Delaunay tiling of  $\mathbb{Z}^n$ . The primitive  $L$ -domain  $\mathcal{D}$  belongs to  $\xi(\text{HYP}_{n+1})$  if and only if its star  $St_0$  contains a main simplex.

A *wall*  $W$  is an  $\frac{n(n+1)}{2} - 1$  dimensional  $L$ -domain, which necessarily separates two primitive  $L$ -domains  $\mathcal{D}$ ,  $\mathcal{D}'$ . Let one moves a point  $a$  from the primitive  $L$ -domain  $\mathcal{D}$  to  $\mathcal{D}'$  by passing through  $W$ . When  $a \in W$ , some pairs of simplices of  $St_0$ , which are mutually adjacent by a facet, glue into repartitioning polytopes. It is well known (see, for example, [DSV06, Vo08]) that an  $n$ -dimensional polytope with  $n + 2$  vertices can be triangulated in exactly two ways. When the point  $a$  goes from  $W$  into the  $L$ -domain  $\mathcal{D}'$ , each repartitioning polytope repartitions into its other set of simplices.

Since each repartitioning polytope has  $n + 2$  vertices, there is an affine dependence between its vertices. This affine dependence generates a linear equality between the coefficients  $a_{ij}$  of the Gram matrix  $a \in \mathcal{S}_{>0}^n$ . This equality is just the equation determining the hyperplane supporting the wall  $W$ . If the point  $a$  lies inside the  $L$ -domain  $\mathcal{D}$ , then this equality holds as an inequality.

So, the convex hull of vertices of any pair of simplices of  $St_0$  adjacent by a facet is a putative repartitioning polytope giving an inequality separating the  $L$ -domain  $\mathcal{D}$  from another  $L$ -domain. All adjacent pairs of simplices of  $St_0$  determine a system of inequalities describing the polyhedral cone of the primitive  $L$ -domain  $\mathcal{D}$ . Note that some of these inequalities define faces of  $\mathcal{D}$  but not walls. If the

n	# primitive $L$ -types	# facets of $\text{HYP}_{n+1}$	# orbits of primitive $L$ -domains in $\text{HYP}_{n+1}$ under $\text{Sym}(n+1)$	# primitive $L$ -domains in $\text{HYP}_{n+1}$
2	1	3	1	1
3	1	12	1	3
4	3	40	5	172
5	222	210	8287	5 338 650

TABLE 2. Decomposition of  $\text{HYP}_{n+1}$  into  $L$ -domains

adjacent pair of simplices contains the main simplex then the corresponding wall lies on a facet of the cone  $\xi(\text{HYP}_{n+1})$ .

Using the above system of inequalities, one can define all extreme rays of  $\mathcal{D}$ , and then all facets of  $\mathcal{D}$ . The sum of Gram matrices lying on extreme rays of  $\mathcal{D}$  is an interior point  $a(\mathcal{D})$  of  $\mathcal{D}$  uniquely related to this  $L$ -domain (see [DSV06] for more details). Hence,  $L$ -domains  $\mathcal{D}$  and  $\mathcal{D}'$  belong to the same  $L$ -type if and only if  $a(\mathcal{D}') = P^T a(\mathcal{D}) P$  for some  $P \in \text{GL}_n(\mathbb{Z})$ , i.e.  $a(\mathcal{D}')$  and  $a(\mathcal{D})$  are arithmetically equivalent.

The algorithm for enumerating primitive  $L$ -domains in  $\xi(\text{HYP}_{n+1})$  works as follows. One takes a primitive Gram matrix  $a \in \xi(\text{HYP}_{n+1})$ . There is a standard algorithm which, for a given  $a \in \mathcal{S}_{>0}^n$ , constructs its simplicial Delaunay tiling. (For example, one can take  $a$  from a principal  $L$ -domain, described in following sections).

Using the star  $St_0$  of the Delaunay tiling, the algorithm, for each pair of adjacent simplices, determines the corresponding inequality. By the system of obtained inequalities, the algorithm finds all extreme rays of the domain  $\mathcal{D}$  of  $a$ , computes the interior central ray  $a(\mathcal{D})$  and finds all facets of  $\mathcal{D}$ . The  $L$ -domain  $\mathcal{D}$  is put in the list  $\mathcal{L}$  of primitive  $L$ -domains in  $\xi(\text{HYP}_{n+1})$ .

Let  $F$  be a facet of  $\mathcal{D}$  which does not lie on a facet of  $\xi(\text{HYP}_{n+1})$ . For each repartitioning polytope related to the facet  $F$  of  $\mathcal{D}$ , the algorithm finds another partition into simplices. This gives the Delaunay tiling of the primitive  $L$ -domain  $\mathcal{D}'$ , which is neighboring to  $\mathcal{D}$  by the facet  $F$ . The algorithm finds all extreme rays of  $\mathcal{D}'$ , the ray  $a(\mathcal{D}')$  and tests if it is arithmetically equivalent to  $a(\mathcal{D})$  for some  $\mathcal{D}$  from  $\mathcal{L}$ . If not, one puts  $\mathcal{D}'$  in  $\mathcal{L}$ . The algorithm stops when all neighboring  $L$ -domains are equivalent to ones in  $\mathcal{L}$ . This algorithm is very similar to the one in [SY04] for the decomposition of the metric cone into  $T$ -domains and belongs to the class of *graph traversal algorithms*.

We give some details of the partition of  $\text{HYP}_{n+1}$  into primitive  $L$ -domains in Table 2.

We now expose another enumeration method of the orbits of primitive  $L$ -domains in  $\text{HYP}_{n+1}$ . Consider a primitive  $L$ -domain  $\mathcal{D}$ . The group  $\text{Stab}(\mathcal{D}) =$

$\{P \in \text{GL}_n(\mathbb{Z}) : P^T a(\mathcal{D})P = a(\mathcal{D})\}$  is a finite group, which permutes the translation classes of simplices of the Delaunay decomposition of  $\mathcal{D}$ . It splits the translation classes of simplices into different orbits. Let  $S$  be a basic simplex in  $\mathcal{D}$ ; if one choose the coordinates such that  $\text{vert } S = \{0, e_1, \dots, e_n\}$  then one obtains an  $L$ -domain  $\mathcal{D}_S$ , whose image by  $\xi^{-1}$  is included in  $\text{HYP}_{n+1}$ . A permutation of the vertex set of  $S$  induces a permutation in  $\text{HYP}_{n+1}$  as well. So, two cones  $\xi^{-1}(\mathcal{D}_S)$  and  $\xi^{-1}(\mathcal{D}_{S'})$  are equivalent under  $\text{Sym}(n+1)$  if and only if  $S$  and  $S'$  belong to the same orbit of translation classes of simplices under  $\text{Stab}(\mathcal{D})$ . Therefore from the list of  $L$ -types in dimension  $n$ , one obtains the orbits of  $L$ -domains in  $\text{HYP}_{n+1}$ .

### 3. DICINGS, RANK 1 EXTREME RAYS OF AN $L$ -DOMAIN AND OF $\text{HYP}_{n+1}$

We denote by  $b^T c$  the scalar product of column vectors  $b$  and  $c$ . A vector  $v \in \mathbb{Z}^n$  is called *primitive* if the greatest common divisor of its coefficients is 1; such a vector defines a family of parallel hyperplanes  $v^T x = \alpha$  for  $\alpha \in \mathbb{Z}$ . In the same way, a *vector family*  $\mathcal{V} = (v_i)_{1 \leq i \leq M}$  of primitive vectors defines  $M$  families of parallel hyperplanes. A vector family  $\mathcal{V}$  is called a *lattice dicing* if for any  $n$  independent vectors  $v_{i_1}, \dots, v_{i_n} \in \mathcal{V}$  the vertices of the hyperplane arrangement  $v_{i_j}^T x = \alpha_j$  form the lattice  $\mathbb{Z}^n$  (see an example on Figure 1). This is equivalent to say that any  $n$  independent vectors  $v_i$  have determinant  $\pm 1$ , i.e. the vector family is *unimodular*. Given a dicing, the connected components of the complement of  $\mathbb{R}^n$  by the hyperplane arrangement form a partition of  $\mathbb{R}^n$  by polytopes. It is proved in [ER94] that the polytopes of a lattice dicing defined by  $\mathcal{V} = (v_i)_{1 \leq i \leq M}$  are Delaunay polytopes for the matrices belonging to the  $L$ -domain generated by the rank 1-forms  $(v_i v_i^T)_{1 \leq i \leq M}$ , whose corresponding quadratic form is  $f_{v_i}(x) = (v_i^T x)^2$ . The reverse is also proved there, i.e. any  $L$ -domain, whose extreme rays have rank 1 has its Delaunay tessellation being a lattice dicing. Such  $L$ -domains are called *dicing domains*; they are simplicial, i.e. their dimension is equal to their number of extreme rays.

Our exposition of matroid theory is limited here to what is useful for the comprehension of the paper and we refer to [DG99, Aig79, Tr92] for more details. Given a graph  $G$  of vertex-set  $\{1, \dots, n\}$  we associate to every edge  $e = (i, j)$  a vector  $v_e$ , which is equal to 1 in position  $i$ ,  $-1$  in position  $j$  and 0 otherwise. The vector family  $\mathcal{V}(G) = (v_e)_{e \in E(G)}$  is unimodular and is called the *graphic unimodular system of the graph  $G$* . Given a unimodular  $n$ -dimensional system  $U$  of  $m$  vectors, for any basis  $B \subseteq U$ , we can write  $U = B(I_n, A)$ , where  $A$  is a totally unimodular matrix and  $(I_n, A)$  is the concatenation of  $I_n$  and  $A$ . The matrix  $(-A^T, I_{m-n})$  defines a unimodular system, which is called the *dual of  $U$*  and denoted  $\text{Dual}(U)$ . Given a graph  $G$  of vertex set  $V(G)$  and edge set  $E(G)$ , we choose an orientation on every edge  $e$  and associate to it a coordinate  $x_e$ . We define a vector space  $V$  to be the set of vector  $v \in \mathbb{R}^{E(G)}$  satisfying for all

$x \in V(G)$  to the *vertex cut equation*

$$0 = \sum_{y \in N(x)} v_{(x,y)} \epsilon_{(x,y)}$$

with  $N(x)$  the neighbors of  $x$  and  $\epsilon_{(x,y)} = 1$  if the orientation of the edge  $(x, y)$  goes from  $x$  to  $y$  and  $-1$  otherwise. Take  $v_1, \dots, v_N$  a basis of the space and denote by  $CoGr(G)$  the *cographic unimodular system of the graph  $G$*  defined to be the vector system obtained by taking the transpose of the matrix  $(v_1, \dots, v_N)$ . The unimodular systems  $CoGr(G)$  and  $Dual(Gr(G))$  are isomorphic. In [DG99] a general method for describing unimodular vector families is given using graphic, cographic unimodular systems and a special unimodular system named  $E_5$  (or  $R_{10}$  as in [Se80]).

Given a finite set  $E$ , a *matroid*  $M = M(E)$  is a family  $\mathcal{C}(M)$  of subset of  $X$  called *circuits* such that:

- for  $C_1, C_2 \in \mathcal{C}(M)$ , it holds  $C_1 \not\subseteq C_2$ ,  $C_2 \not\subseteq C_1$  if  $C_1 \neq C_2$ ;
- if  $e \in C_1 \cap C_2$ , then there is  $C_3 \in \mathcal{C}(M)$  such that  $C_3 \subseteq C_1 \cup C_2 - \{e\}$ .

A set of vectors  $q_e$ ,  $e \in E$ , represents a matroid  $M(E)$  if, for any circuit  $C \in \mathcal{C}(M)$ , the equality  $\sum_{e \in C} q_e = 0$  holds. A matroid, is called *regular* if it admits a representation as a unimodular system of vectors. If  $M(E)$  is a *graphic* or *cographic* matroid of a graph  $G$  with a set  $E$  of edges, then circuits of  $M$  are *cycles* or *cuts* of  $G$ , respectively.

A graph  $G$  is *plane* if it is embedded in the 2-plane such that any two edges are non-crossings. A plane graph defines a partition of the plane into faces delimited by edges. The *dual graph*  $G^*$  is the graph defined by faces with an edge between two faces if they share an edge. Then  $(G^*)^* = G$ , and there is a bijection between (intersecting) edges of  $G$  and  $G^*$  such that each cut of  $G$  corresponds to a cycle of  $G^*$ , and vice versa. In other words, the cographic matroid of  $G$  and the graphic matroid of  $G^*$  are isomorphic.

The only rank 1 extreme rays of the cone  $HYP_{n+1}$  are *cut metrics*. For  $n \leq 5$ , the hypermetric cone  $HYP_{n+1}$  coincides with the cut cone  $CUT_{n+1}$  which is the cone hull of  $2^n - 1$  cut metrics. Denote  $N = \{1, \dots, n\}$ ; if  $S \subset N$ ,  $S \neq \emptyset$  then the cut metric  $\delta_S$  on  $X = \{0\} \cup N$  is defined as follows:

$$\delta_S(i, j) = 1 \text{ if } |\{ij\} \cap S| = 1, \text{ and } \delta_S(i, j) = 0, \text{ otherwise.}$$

The covariance map  $\xi$  transforms the cut metric  $\delta_S$  into the following *correlation* matrix  $p(S)$  of rank 1:

$$p_{ij}(S) = 1 \text{ if } \{ij\} \subset S, \text{ and } p_{ij}(S) = 0, \text{ otherwise, where } 1 \leq i, j \leq n.$$

The quadratic form corresponding to the correlation matrix  $p(S)$  is

$$f_S(x) = \xi(\delta_S)(x) = \sum p_{ij}(S) x_i x_j = \left( \sum_{i \in S} x_i \right)^2.$$



So,  $f_S = f_q$  with  $f_q(x) = (q^T x)^2$  and  $q = \sum_{i \in S} b_i := b(S)$  the incidence vector of the set  $S$ . In summary:

**Lemma 1.** *A vector  $q \in \mathbb{Z}^n$  determines an extreme ray  $f_q$  of  $\xi(\text{HYP}_{n+1})$  if and only if  $q = b(S)$  for some  $S \subseteq N$ ,  $S \neq \emptyset$ .*

By Lemma 1, if  $U$  determines a dicing domain in  $\xi(\text{HYP}_n)$ , then the set of coordinates of vectors from  $U$  in the basis  $\mathcal{B}$  forms a unimodular matrix with  $(0, 1)$ -coefficients. Note that the columns of any  $(0, 1)$ -matrix are incidence vectors of subsets of a set. Since any unimodular system  $U$  determines a dicing  $L$ -domain  $\mathcal{D}(U)$ , we have the following proposition:

**Proposition 1.** *Let  $\mathcal{D}(U)$  be a dicing domain determined by a unimodular set  $U$ . The following assertions are equivalent:*

- (i)  $\mathcal{D}(U)$  lies in  $\xi(\text{HYP}_n)$ ;
- (ii)  $U$  is represented by a  $(0, 1)$ -matrix.

#### 4. THE PRINCIPAL $L$ -DOMAIN

There is a unique primitive  $L$ -type, whose  $L$ -domains are dicing domains of maximal dimension  $\frac{1}{2}n(n+1)$  ([KoZo77, Di72]). Voronoi calls this  $L$ -type *principal*. Each principal  $L$ -domain is simplicial and all its  $\frac{1}{2}n(n+1)$  extreme rays have rank 1. The set  $Q$  of vectors  $q$  determining extreme rays  $f_q$  of a principal  $L$ -domain forms a maximal unimodular system. This system is the classical unimodular root system  $A_n$  representing the graphic matroid of the complete graph  $K_{n+1}$  on  $n+1$  vertices.

In our case, when a principal  $L$ -domain is contained in  $\xi(\text{HYP}_{n+1})$ , its extreme rays belong to the set  $\{p(S) : S \subseteq N, S \neq \emptyset\}$  of extreme rays of the cone  $\xi(\text{HYP}_{n+1})$ . Hence, the vectors  $q$  have the form  $b(S)$  for  $S \subseteq N$ . We shall find all subsets of these vectors representing the graphic matroid of  $K_{n+1}$ . We orient edges of  $K_{n+1}$  into *arcs* and relate a vector  $b(S)$  to each arc of the directed graph  $K_{n+1}$  such that, for any directed circuit  $C$  in  $K_{n+1}$ , the following equality holds

$$(2) \quad \sum_{e \in C} \varepsilon_e b(S_e) = 0.$$

Here  $b(S_e)$  is the vector related to the arc  $e$  and  $\varepsilon_e = 1$  if the directions of  $e$  and  $C$  coincide, and  $\varepsilon_e = -1$ , otherwise.

Given a chain of equally directed arcs labeled by one-element set, a subchain of this chain determines a  $(0, 1)$ -characteristic vector. It is known that the set of characteristic vectors of a set of connected subchains determines a graphic unimodular system. We show below that such graphical systems are contained in the set  $\{b(S) : S \subseteq N\}$ .

This relation of vectors  $b(S)$  and arcs of  $K_{n+1}$  provides a labeling of arcs of  $K_{n+1}$  by subsets  $S \subseteq N$ . We call this labeling *feasible* if the corresponding set



of vectors  $b(S)$  gives a representation of the graphical matroid of  $K_{n+1}$ , i.e. (2) holds for each circuit  $C$  of  $K_{n+1}$ .

Consider a  $k$ -circuit  $C = \{e_i : 1 \leq i \leq k\}$ , whose arcs have the same directions. Suppose that, for  $1 \leq i \leq k$ ,  $S_i$  is a label of the arc  $e_i$ , and that this labeling is feasible. Then the equality  $\sum_{i=1}^k b(S_i) = 0$  holds. Since the coordinates of the vectors  $b(S)$  take  $(0, 1)$ -values, this equality is not possible for a feasible labeling. Hence, the directed graph  $K_{n+1}$  with a feasible labeling has no circuit, whose arcs have the same directions. Any finite directed graph with no circuit has at least one *source* vertex (and a *sink* vertex as well).

Now consider a directed 3-circuit  $C = \{e_1, e_2, e_3\}$  of a feasible labeled  $K_{n+1}$ . Then two arcs of  $C$ , say the arcs  $e_1, e_2$ , have directions coinciding with the direction of  $C$ , and the third arc  $e_3$  has opposite direction. If  $S_i$  is a label of  $e_i$ ,  $i = 1, 2, 3$ , then we have the equality  $b(S_1) + b(S_2) = b(S_3)$ . This equality is possible only if  $S_1 \cap S_2 = \emptyset$  and  $S_1 \cup S_2 = S_3$ . Since any two adjacent arcs of a complete graph belong to a 3-circuit, we obtain the following result:

**Lemma 2.** *Let two arcs  $e_i$  and  $e_j$  be adjacent in a feasible labeled graph  $K_{n+1}$  and have labels  $S_i$  and  $S_j$ . Then  $S_i \cap S_j = \emptyset$  if the directions of these arcs coincide in the 2-path  $[e_i, e_j]$ . If these directions are opposite, then either  $S_i \subset S_j$  or  $S_j \subset S_i$ .*

Let  $v$  be a source vertex of  $K_{n+1}$  and  $E(v)$  be the set of  $n$  arcs incident to  $v$ . Since all arcs of  $E(v)$  go out from  $v$ , any two arcs  $e, e' \in E(v)$  have opposite directions in their 2-path  $[e, e']$ . Let  $\mathcal{S}(v) = \{S_e : e \in E(v)\}$ . By Lemma 2, the family  $\mathcal{S}(v)$  is a nested family of  $n$  mutually embedded distinct subsets. This implies that the sets  $S \in \mathcal{S}(v)$  and the arcs  $e \in E(v)$  can be indexed as  $S_i, e_i$ ,  $1 \leq i \leq n$ , such that  $S_i$  is the label of  $e_i$  and  $|S_i| = i$ .

For  $2 \leq i \leq n$ , let  $g_i$  be the arc of the graph  $K_{n+1}$ , which forms a 3-circuit with the arcs  $e_{i-1}, e_i \in E(v)$ . Lemma 2 implies that the arc  $g_i$  has the one-element set  $S_i - S_{i-1}$  as label, and the direction of  $g_i$  coincides with the direction of  $e_{i-1}$  in their 2-path  $[e_{i-1}, g_i]$ . Now, it is clear that the  $n$  arcs  $g_i$  for  $1 \leq i \leq n$ , where  $g_1 = e_1$ , form an  $n$ -path, whose arcs have the same directions and are labeled by one-element sets. Recall that a non self-intersecting  $n$ -path in a graph with  $n+1$  vertices is called a *Hamiltonian path*. We obtain the following result.

**Lemma 3.** *A feasible labeled complete directed graph  $K_{n+1}$  has a Hamiltonian path such that all its arcs have the same directions and each arc has a one-element labeling set.*

Let  $\{0\} \cup N$  be the set of vertices of  $K_{n+1}$ , where the vertex 0 is the source. Let  $0, i_1, i_2, \dots, i_n$  be the vertices of the Hamiltonian path  $\pi$  in Lemma 3. The path  $\pi$  defines uniquely an orientation and a feasible labeling of  $K_{n+1}$  as follows. The arc with end-vertices  $i_j, i_k$ , where  $0 \leq j < k \leq n$ , is labeled by the set  $S_{jk} = \{i_r : j+1 \leq r \leq k\} \subseteq N$ . If one reverse the above order, then one gets the same family of sets, and the labeled graph  $K_{n+1}$  gives the same representation of the unimodular system  $A_n$ . We have

**Lemma 4.** *Any representation of the graphic matroid of the complete graph  $K_{n+1}$  by vectors  $b(S)$ ,  $S \subseteq N$ ,  $S \neq \emptyset$ , is determined by a complete order of the set  $N$ . Two opposite orders determine the same representation.*

Since there are  $n!$  complete orders on an  $n$ -set, as a corollary of Lemma 4, we obtain our main result.

**Proposition 2.** *The cone  $\xi(\text{HYP}_{n+1})$  contains  $\frac{1}{2}n!$  distinct principal  $L$ -domains.*

So, each principal domain is determined by an order (and its reverse) of the set  $N$ . For the sake of definition, we choose the lexicographically minimal order  $\mathcal{O}$  from these two orders. Let  $\mathcal{S}(\mathcal{O})$  be the family of sets  $S \subseteq N$ ,  $S \neq \emptyset$ , such that elements of each set  $S$  determine a continuous subchain of the  $n$ -chain, corresponding to the order  $\mathcal{O}$ . A principal domain determined by an order  $\mathcal{O}$  of the set  $N$  has  $\frac{1}{2}n(n+1)$  extreme rays  $p(S)$  for  $S \in \mathcal{S}(\mathcal{O})$ .

Each face  $F$  of a dicing  $L$ -domain in  $\xi(\text{HYP}_{n+1})$  is uniquely determined by its extreme rays  $p(S)$ , all of rank 1. Set

$$\mathcal{S}(F) = \{S \subseteq N : p(S) \text{ is an extreme ray of } F\}.$$

**Proposition 3.** *If  $n \geq 4$ , then any two principal domains in  $\xi(\text{HYP}_{n+1})$  are not contiguous by a facet.*

*Proof.* If two principal domains  $\mathcal{D}(\mathcal{O}), \mathcal{D}(\mathcal{O}') \subset \xi(\text{HYP}_{n+1})$  share a facet  $F$ , then they have  $\frac{1}{2}n(n+1) - 1$  common extreme rays  $p(S)$  for  $S \in \mathcal{S}(F)$ . This implies that the families  $\mathcal{S}(\mathcal{O})$  and  $\mathcal{S}(\mathcal{O}')$  should differ by one element only. But, for any two distinct orders  $\mathcal{O}$  and  $\mathcal{O}'$ , the families  $\mathcal{S}(\mathcal{O})$  and  $\mathcal{S}(\mathcal{O}')$  differ at least by two sets, if  $n \geq 4$ . For example, suppose  $\mathcal{O}$  and  $\mathcal{O}'$  differ by a transposition of two elements  $i$  and  $j$ . Then there is at least one subchain in  $\mathcal{O}$  containing  $i$  and not containing  $j$ , which is not a subchain of  $\mathcal{O}'$ . The same assertion is true for the order  $\mathcal{O}'$ .  $\square$

## 5. THE DECOMPOSITIONS OF $\text{HYP}_3$ AND $\text{HYP}_4$

Since, for  $n = 2$  and  $n = 3$  there exists only one primitive  $L$ -type, namely, the principal  $L$ -type, Proposition 2 describes completely the decompositions of the cones  $\xi(\text{HYP}_{n+1})$  for  $n = 2, 3$ .

Recall that each facet  $F$  of  $\text{HYP}_{n+1}$  is described by an inequality (1). If  $n \leq 4$  and  $F$  is a facet of  $\text{HYP}_{n+1}$ , then  $z_i \in \{0, \pm 1\}$ . Hence, we can denote triangle and pentagonal facets as  $F(ij; k)$  and  $F(ijk; lm)$ , respectively. Here  $z_i = z_j = 1$ ,  $z_k = -1$ ,  $z_l = 0$ ,  $l \in X - \{ijk\}$ , for the triangle facet, and  $z_i = z_j = z_k = 1$ ,  $z_l = z_m = -1$ ,  $z_r = 0$  if  $r \in X - \{ijklm\}$ , for the pentagonal facet.

Note that  $\xi(\delta_S) = p(S)$ . For  $S = \{ij \dots k\}$ , set  $S = ij \dots k$  and  $p(S) = p(ij \dots k)$ .

**$n = 2$ .** The cone  $\xi(\text{HYP}_3)$  is three-dimensional and simplicial. There is only one order  $\mathcal{O} = (12)$  with  $\mathcal{S}(\mathcal{O}) = \{1, 2, 12\}$ . Its three extreme rays  $p(1)$ ,  $p(2)$  and  $p(12)$  span a principal domain, i.e.  $\xi(\text{HYP}_3)$  coincides with a principal domain.

$\mathbf{n} = \mathbf{3}$ . The cone  $\xi(\text{HYP}_4)$  is six-dimensional and has seven extreme rays  $p(i)$ ,  $i = 1, 2, 3$ ,  $p(ij)$ ,  $ij = 12, 13, 23$ ,  $p(123)$  and 12 facets  $\xi(F(ij; k))$  for  $i, j, k \in \{0\} \cup N$ . Note that

$$p(1) + p(2) + p(3) + p(123) = p(12) + p(13) + p(23) = p.$$

Hence, the four-dimensional cone  $\mathcal{C}_1 = \mathbb{R}_+p(1) + \mathbb{R}_+p(2) + \mathbb{R}_+p(3) + \mathbb{R}_+p(123)$  intersects by the ray  $\mathbb{R}_+p$  the three-dimensional cone  $\mathcal{C}_2 = \mathbb{R}_+p(12) + \mathbb{R}_+p(13) + \mathbb{R}_+p(23)$ . The ray  $\mathbb{R}_+p$  is an interior ray of both.

By Proposition 2,  $\xi(\text{HYP}_4)$  contains three principal domains. These three six-dimensional  $L$ -domains are determined by the three orders (123), (132) and (213) of  $N = \{123\}$ . Denote the domain determined by the order  $(ijk)$  as  $\mathcal{D}_j$ , where  $j$  is the middle element of the order  $(ijk)$ . Since the three one-element subsets  $\{i\}$ ,  $i \in N$ , and the set  $N$  give continuous chains in all the three orders, the four rays  $p(1), p(2), p(3)$  and  $p(123)$  are common rays of all the three domains  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$ . Hence, the cone  $\mathcal{C}_1$  is the common four-dimensional face of these three principal domains. The domain  $\mathcal{D}_i$  is the cone hull of  $\mathcal{C}_1$  and two rays  $p(ij)$  and  $p(ik)$ . The four triangle facets  $\xi(F(jk; i))$ ,  $\xi(F(jk; 0))$ ,  $\xi(F(0i; j))$ ,  $\xi(F(0i; k))$  of  $\xi(\text{HYP}_4)$  are also facets of  $\mathcal{D}_i$ . The other two facets of  $\mathcal{D}_i$  separating the domain  $\mathcal{D}_i$  from  $\mathcal{D}_j$  and  $\mathcal{D}_k$  are the cone hulls of  $\mathcal{C}_1$  with the rays  $p(ij)$  and  $p(ik)$ , respectively.

## 6. $L$ -DOMAINS IN $\xi(\text{HYP}_5)$

A *parallelohedron* is an  $n$ -dimensional polytope, whose image under a translation group forms a tiling of  $\mathbb{R}^n$ . Given a face  $F$  of a parallelohedron  $P$ , the set of faces of  $P$  which are translates of  $F$  is called the *zone* of  $P$ . For a parallelohedron  $P$  the Minkowski sum  $P + z(q)$  may not be a parallelohedron. A parallelohedron  $P$  is called *free along a vector*  $q$  and the vector  $q$  is called *free for a parallelohedron*  $P$  if the sum  $P + z(q)$  is a parallelohedron (see [Gr06] for more details on this notion).

If  $P_q = P + z(q)$  is an  $n$ -dimensional parallelohedron, then  $P_q$  has a *non-zero width* along the line  $l(q)$  spanned by  $q$ . This means that the intersection of  $P_q$  with a line parallel to  $l(q)$  is distinct from a point. In this case, the lattice  $L_q$  of the parallelohedron  $P_q$  has a *lamina*  $H$ , i.e. a hyperplane  $H$  such that  $H$  is transversal to  $l(q)$ , the intersection  $L_q \cap H$  is an  $(n - 1)$ -dimensional sublattice of  $L_q$  and each Delaunay polytope of  $L_q$  lies between two neighbouring layers of  $L_q$  parallel to  $L_q \cap H$  (see [DeG02]). If a Voronoi polytope has a non-zero width along a line  $l$ , then the lamina  $H$  is orthogonal to  $l$ .

**6.1. Root lattice  $D_4$ .** The lattice  $D_n$  is defined as

$$D_n = \{x \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i \equiv 0 \pmod{2}\}$$

If  $\{e_i : 1 \leq i \leq n\}$  is an orthonormal basis of  $\mathbb{Z}^n$ , then the set of shortest vectors of  $\mathbf{D}_n$  is  $\pm e_i \pm e_j$  for  $1 \leq i < j \leq n$ . It is the set of all facet vectors of  $P_V(\mathbf{D}_n)$  and form an irreducible root system, which we also denote by  $\mathbf{D}_n$ . There are three translation classes of Delaunay polytopes in  $\mathbf{D}_n$ : the cross polytope  $\beta_n$  whose vertex set is formed by all  $e_1 \pm e_i$  for  $1 \leq i \leq n$ , the half cube  $\frac{1}{2}H_n$  whose vertex set is  $\{x \in \{0, 1\}^n \mid \sum_{i=1}^n x_i \equiv 0 \pmod{2}\}$  and a second half cube  $\frac{1}{2}H'_n$  whose vertex set is  $\{x \in \{1, 2\} \times \{0, 1\}^{n-1} \mid \sum_{i=1}^n x_i \equiv 0 \pmod{2}\}$ . The two half cubes are equivalent under the automorphism group  $\text{Aut}(\mathbf{D}_n)$  of the root lattice  $\mathbf{D}_n$ . It is proved in [Gr04, DeG03] that the Voronoi polytope  $P_V(\mathbf{D}_n)$  of the  $n$ -dimensional root lattice  $\mathbf{D}_n$  is free only along vectors which are parallel to edges of  $P_V(\mathbf{D}_n)$ . Thus there are  $2^n + 2n$  free vectors.

It turns out that when  $n = 4$  all Delaunay polytopes are isometric to the cross-polytope  $\beta_4$  and equivalent under  $\text{Aut}(\mathbf{D}_4)$ . Any 2-face of  $\beta_4$  is contained in three Delaunay polytopes  $\beta_4$  of the Delaunay tessellation. This proves that the  $L$ -domains of  $\mathbf{D}_4$  are not dicing domains. Furthermore,  $\mathbf{D}_4$  is a *rigid lattice* (see [BaGr01]), i.e. its Delaunay tessellation determines the Gram matrix up to a scalar multiple. This means that  $\mathbf{D}_4$  determine a 1-dimensional  $L$ -type and that any  $L$ -domain containing it as an extreme ray is not a dicing domain.

The free vectors of  $P_V(\mathbf{D}_4)$  are parallel and can be identified to the diagonals of the cross polytopes. They are

- $\pm 2e_i$  with  $1 \leq i \leq 4$  for  $\beta_4$
- $(\pm 1, \pm 1, \pm 1, \pm 1)$  with even plus signs for  $\frac{1}{2}H_4 \equiv \beta_4$
- $(\pm 1, \pm 1, \pm 1, \pm 1)$  with odd plus signs for  $\frac{1}{2}H'_4 \equiv \beta_4$

Up to a factor  $\sqrt{2}$ , those 24 vectors are an isometric copy of the root system  $\mathbf{D}_4$ , which we denote by  $\mathbf{D}_{4,2}$ . The union  $\mathbf{D}_4 \cup \mathbf{D}_{4,2}$  is the irreducible root system  $\mathbf{F}_4$  (see [Hu90]).

The Voronoi polytope  $P_V(\mathbf{D}_4)$  of the root lattice  $\mathbf{D}_4$  is the regular polytope 24-cell, whose automorphism group is the Coxeter group  $W(\mathbf{F}_4)$  of Schläfli symbol  $\{3, 4, 3\}$  (see [Cox48]). Its number of vertices, 2-faces, 3-faces is 24, 96, 24. Each facet is an octahedron with four pairs of opposite and mutually parallel triangular 2-faces. The facet vectors of the Voronoi polytope  $P_V(\mathbf{D}_4)$  are the 24 roots of the root system  $\mathbf{D}_4$ . The polytope  $P_V(\mathbf{D}_4)$  has 12 edge *zones* of mutually parallel edges representing  $\mathbf{D}_{4,2}$  and 16 face zones of mutually parallel triangular faces. Each edge zone contains 8 parallel edges, and each face zone contains six parallel faces.

We choose a basis of  $\mathbf{D}_4$  and denote by  $a(\mathbf{D}_4)$  the Gram matrix of  $\mathbf{D}_4$  in this basis. The 24 vertices of  $P_V(\mathbf{D}_4)$  are given by  $\frac{1}{2}\mathbf{D}_{4,2}$ .

**6.2. The  $L$ -domains containing  $a(\mathbf{D}_4)$ .** For  $n = 4$ , there are three primitive  $L$ -types of four-dimensional lattices: the principal type, and  $L$ -types called by Delaunay in [De29] types II and III. The ten-dimensional  $L$ -domains of these  $L$ -types are constructed as follows (cf., [DeG03]).

Each  $k$ -dimensional face of a principal domain relates to the graphic matroid of a subgraph  $G \subseteq K_5$  on  $k$  edges. Hence each facet (of dimension 9) of a principal domain relates to the graphic matroid of  $K_5 - \mathbf{1}$ , i.e. the complete graph  $K_5$  without one edge. The Gram matrix  $a(D_4)$  of the root lattice  $D_4$  is an extreme ray of  $L$ -domains of type II and III. The cone hull of a facet of a principal domain and of a ray of type  $a(D_4)$  is an  $L$ -domain of type II. Hence, any principal domain is contiguous in  $\mathcal{S}_{>0}^4$  by facets only with  $L$ -domains of type II.

An  $L$ -domain of type II has the following three types of facets. One dicing facet by which it is contiguous in  $\mathcal{S}_{>0}^4$  to a principal domain relates to the graphic matroid of the graph  $K_5 - \mathbf{1}$ . Each of two other types of facets is the cone hull of the ray of type  $a(D_4)$  and a dicing 8-dimensional face related to the graphic matroids of  $K_5 - 2 \times \mathbf{1}$  or of  $K_5 - \mathbf{2}$ . Here each of these graphs is the complete graph  $K_5$  without two non-adjacent or two adjacent edges, respectively.

The complete bipartite graph  $K_{ij}$  is formed by two blocks  $S_1, S_2$  of vertices with  $|S_1| = i, |S_2| = j$  and two vertices adjacent if and only if they belong to different blocks. An  $L$ -domain of type III is the cone hull of  $a(D_4)$  and a 9-dimensional dicing facet related to the cographic matroid  $CoGr(K_{33})$  of the bipartite graph  $K_{33}$ . Each 8-element submatroid of  $CoGr(K_{33})$  is graphic and relates to the graph  $K_5 - 2 \times \mathbf{1}$ . Hence, each other facet of an  $L$ -domain of type III is the cone hull of  $a(D_4)$  and a dicing 8-dimensional face related to  $K_5 - 2 \times \mathbf{1}$ . In  $\mathcal{S}_{>0}^4$ , this facet is a common facet of  $L$ -domains of types II and III.

So, an  $L$ -domain of type II is contiguous in  $\mathcal{S}_{>0}^4$  to  $L$ -domains of all three types. It is useful to note ([RB05, DV05]) that if  $f$  belongs to the closure  $\mathcal{D}$  of an  $L$ -domain of type II or III then the Voronoi polytope  $P_V(f)$  of  $\mathbb{Z}^n$  under the quadratic form  $f$  is an affine image of the Minkovski sum

$$\sum_{q \in U} \lambda_q z(q) + \lambda P_V(D_4), \quad \lambda_q \geq 0, \lambda \geq 0,$$

where  $U$  is the unimodular set of vectors related to rank 1 extreme rays of  $\mathcal{D}$ , and  $P_V(D_4)$  is the Voronoi polytope of the root lattice  $D_4$ , whose form  $a(D_4)$  lies also on extreme ray of  $\mathcal{D}$ .

**6.3. Unimodular systems in  $D_{4,2}$ .** Let  $D(4) \subset D_{4,2}$  be a subset of 12 roots chosen by one from each pair of opposite roots. The vector system  $D(4)$  is partitioned into three disjoint *quadruples*  $Q_i, i = 1, 2, 3$ , of mutually orthogonal roots, given in Subsection 6.1.

For what follows, we have to consider *triples*  $(r_1, r_2, r_3)$  of roots chosen by one from each quadruple  $Q_i$ , i.e.  $r_i \in Q_i, i = 1, 2, 3$ . Let  $t = (r_1, r_2, r_3)$  be such a triple. Let  $\{ijk\} = \{123\}$ , i.e. these three indices are distinct. We have  $r_i^2 = 2, r_i^T r_j \in \{\pm 1\}$ . Hence, the vectors

$$(3) \quad r_{ij} = r_i - (r_i^T r_j) r_j \text{ for } ij = 12, 23, 31,$$

are roots of  $D_{4,2}$ . Since  $r_{ij}^T r_i = -r_{ij}^T r_j = 1$ , one of two opposite roots  $\pm r_{ij}$  belongs to  $Q_k$ , say  $r_{ij} \in Q_k$ . Hence  $r_{12} \in Q_3, r_{23} \in Q_1, r_{31} \in Q_2$ . We have two cases:

- (i)  $r_{ij}$  belongs, up to sign, to the triple  $t$ , i.e.  $r_{ij} = r_k$ , for all pairs  $ij$ ;
- (ii)  $r_{ij}$  does not belong to  $t$ , i.e.  $r_{ij} \neq r_k$ .

In case (i), the vectors  $r_i$ ,  $i = 1, 2, 3$ , are linearly dependent, and the triple  $t$  spans a 2-dimensional plane. We say that the triple  $t$  is of *rank 2*. Note that any two roots  $r, r'$  from distinct quadruples determine uniquely the third root  $r'' = r - (r^T r')r'$  and that there are 16 distinct triples of rank 2.

In case (ii), the roots  $r_i$  of the triple  $t$  are linearly independent. We say that the triple  $t$  has rank 3. In this case the roots  $r_{ij}$  for  $ij = 12, 23, 31$ , are distinct and do not coincide with the roots  $r_i$ ,  $i = 1, 2, 3$ . Moreover, it is not difficult to verify that the triple  $(r_{12}, r_{23}, r_{31})$  has rank 2.

Triples of rank 2 and 3 are realized in  $\sqrt{2}P_V(D_4)$  as follows. The roots of a triple  $t$  of rank 2 are parallel to edges of a 2-face of  $\sqrt{2}P_V(D_4)$ . We say that a triple  $t$  of rank 2 forms a face of  $\sqrt{2}P_V(D_4)$ . Since a triple of rank 2 forms a face of  $\sqrt{2}P_V(D_4)$ , the 16 triples of rank 2 relate to the 16 zones of triangular faces of it.

Let  $t' = (r'_1, r'_2, r'_3)$  be a triple of rank 3. The 6 vectors  $\pm r'_i$ ,  $i = 1, 2, 3$ , have end-vertices in vertices of  $\sqrt{2}P_V(D_4)$ . From each pair  $\pm r'_i$  of opposite roots, we choose a vector  $r_i$  such that  $r_i^T r_j = 1$  for  $ij = 12, 23, 31$ . Then end-vertices of the roots  $r_i$  are vertices of a face  $F$  of  $\sqrt{2}P_V(D_4)$ , and the triple  $t = (r_{ij} = r_i - r_j : ij = 12, 23, 31)$  has rank 2 and forms the face  $F$ .

For what follows, we need subsets  $U$  of  $D(4)$  which are maximal by inclusion such that  $P_V(D_4) + \sum_{q \in U} \lambda_q z(q)$  is a parallelohedron. Note that each quadruple  $Q_i$  is a basis of  $\mathbb{R}^4$ . Obviously, it is a unimodular set. It is a maximal by inclusion unimodular subset of  $D(4)$ , since any other vector of  $D(4)$  has half-integer coordinates in this basis. However, it is proved in [DeG03], that  $P_V(D_4) + \sum_{q \in U} \lambda_q z(q)$  is not a parallelohedron if  $U \subset D(4)$  is a quadruple, and it is a parallelohedron for any other maximal unimodular subsets  $U \subseteq D(4)$ . Of course, a maximal unimodular set  $U \neq Q_i$  does not contain each quadruple  $Q_i$  as a subset.

We show below that maximal unimodular subsets in  $D(4)$  represent either the graphic matroid of the graph  $K_5 - 1$  or the cographic matroid of the graph  $K_{33}$  (see Figure 2).

The graph  $K_5 - 1$  is planar. Hence, the graphic matroid of  $K_5 - 1$  is isomorphic to the cographic matroid of the graph  $(K_5 - 1)^*$ . Both graphs  $(K_5 - 1)^*$  and  $K_{33}$  are cubic graphs on six vertices and nine edges. These graphs have a Hamiltonian 6-circuit  $C_6$  on six vertices  $v_i$ ,  $1 \leq i \leq 6$ . Call  $C_6$  with the six edges  $(v_i, v_{i+1})$  by a *rim*, and the other three edges by *spokes*. The spokes are the following edges:  $e_1 = (v_1, v_4)$ ,  $e_2 = (v_2, v_6)$ ,  $e_3 = (v_3, v_5)$  in  $(K_5 - 1)^*$ , and  $e_i = (v_i, v_{i+3})$ ,  $i = 1, 2, 3$ , in  $K_{33}$  (see Figure 2). Note that the edges  $e_2$  and  $e_3$  do not intersect in  $(K_5 - 1)^*$  and intersect in  $K_{33}$ . The described form of  $K_{33}$  is the graph  $Q_4$  of [DG99].

Besides the planarity, the graph  $(K_5 - 1)^*$  differs from  $K_{33}$  by the number of cuts of cardinality three. All cuts of cardinality 3 of  $K_{33}$  are the six one-vertex cuts,



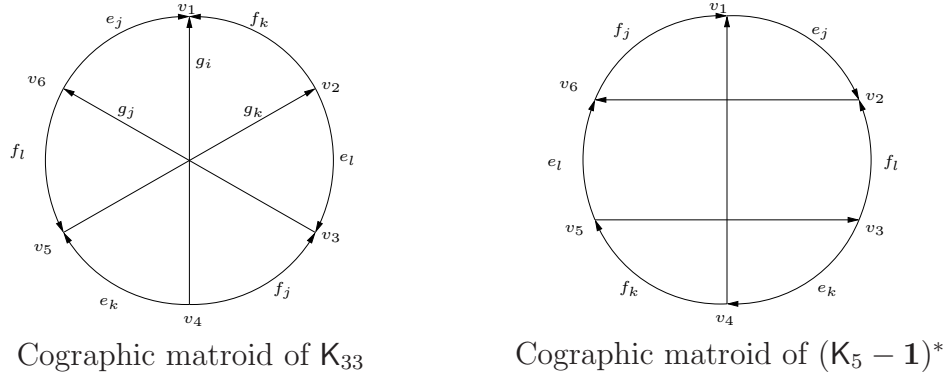


FIGURE 2. Two cographic matroids

i.e. cuts containing three edges incident to a vertex. The graph  $(K_5 - 1)^*$ , besides the six one-vertex cuts, has a separating cut containing three non-adjacent edges one of which is a spoke. (These are the edges  $(v_2, v_3)$ ,  $(v_5, v_6)$  and the spoke  $(v_1, v_4)$  of the above description of  $(K_5 - 1)^*$ .)

**Proposition 4.** *A maximal by inclusion unimodular subset  $U$  of  $D(4)$ ,  $U \neq Q_i$ ,  $i = 1, 2, 3$ , is obtained by a deletion from  $D(4)$  of a triple  $t = (r_1, r_2, r_3)$  where  $r_i \in Q_i$ ,  $i = 1, 2, 3$ . Then*

- (i) *if  $t$  has rank 2, then  $U$  represents the cographic matroid of the graph  $K_{33}$ ;*
- (ii) *if  $t$  has rank 3, then  $U$  represents the graphic matroid of the graph  $K_5 - 1$ , which is isomorphic to the cographic matroid of the dual graph  $(K_5 - 1)^*$ .*

*Proof.* Since  $Q_i \not\subseteq U$ , for  $i = 1, 2, 3$ ,  $U$  does not contain a triple  $t$ . We show that a deletion from  $D(4)$  of any triple gives a unimodular set.

Denote by  $V(r)$  the set of all triples of rank 2 containing a root  $r \in D(4)$ . If  $r \in Q_i$  and  $r' \in Q_j$ , then  $V(r) \cap V(r')$  is the unique triple of rank 2 containing  $r$  and  $r'$  if  $i \neq j$ , and  $|V(r) \cap V(r')| = 0$  if  $i = j$  and  $r \neq r'$ .

Let  $r_i \in Q_i$ ,  $i = 1, 2, 3$ , be the roots of the deleted triple  $t$ . Then

$$|V(r_i)| = 4, |V(r_i) \cap V(r_j)| = 1, \text{ and } |V(r_1) \cap V(r_2) \cap V(r_3)| = 0 \text{ or } 1.$$

Here, whether 0 or 1 stays in the last equality depends on the triple  $t$  has rank 3 or 2, respectively. By the inclusion-exclusion principle, we have

$$|\cup_{i=1}^3 V(r_i)| = \sum_{i=1}^3 |V(r_i)| - \sum_{1 \leq i < j \leq 3} |V(r_i) \cap V(r_j)| + |\cap_{i=1}^3 V(r_i)|.$$

Hence, we have the following two cases.

- (i) If  $|\cap_{i=1}^3 V(r_i)| = 1$ , i.e. if the triple  $t$  has rank 2, then  $|\cup_{i=1}^3 V(r_i)| = 3 \cdot 4 - 3 \cdot 1 + 1 = 10$ .
- (ii) If  $|\cap_{i=1}^3 V(r_i)| = 0$ , i.e. if the triple  $t$  has rank 3, then  $|\cup_{i=1}^3 V(r_i)| = 9$ .



Let  $V = \cup_{r \in D(4)} V(r) - \cup_{i=1}^3 V(r_i)$ . Then  $V$  is the set of triples of rank 2 that do not contain the deleted roots  $r_1, r_2$  and  $r_3$ . Using the set  $V$  we construct a cubic directed graph  $G$  as follows. We take  $R_0 = D(4) - t$  to be the set of 9 arcs of  $G$ . As a set of vertices of  $G$  we take a subset  $V_0 \subseteq V$  of six triples. An arc  $r \in R_0$  is incident to a vertex  $v \in V_0$  if the root  $r$  belongs to the triple  $v$  of rank 2. Hence, each vertex  $v$  is incident to three edges, and the graph  $G$  is cubic. Directions of arcs are chosen as follows. Let the roots  $r, s, p$  of a triple  $v$  satisfy the equation  $r - (r^T s)s - (r^T p)p = 0$ . Suppose that the arc  $r$  comes in  $v$ . Then the arc  $s$  goes out or comes in the vertex  $v$  if  $r^T s = 1$  or  $r^T s = -1$ , respectively. The same assertion is true for the arc  $p$ .

We show that each root  $r \in R_0$  belongs to exactly two triples of  $V_0$ . Let  $r \in Q_k$ , then  $|V(r) \cap V(r_k)| = 0$  and  $|V(r) \cap V(r_i)| = |V(r) \cap V(r_j)| = 1$ , where  $\{ijk\} = \{123\}$ . We have two cases:

$$(4) \quad V(r) \cap V(r_i) = V(r) \cap V(r_j) \text{ or } V(r) \cap V(r_i) \neq V(r) \cap V(r_j).$$

Suppose that the inequality in (4) holds for  $r$ . Since  $|V(r)| = 4$ , only two triples from  $V(r)$  belong to  $V$ .

Now, if the equality in (4) holds, then it implies that  $r = (r^T r_i)r_i + (r^T r_j)r_j$ . If  $t$  has rank 2, then, up to sign, this gives  $r = r_k$ , which contradicts to  $r \in R_0$ . Hence, if  $t$  has rank 2, the inequality of (4) holds for all  $r \in R_0$ . We can set  $V_0 = V$ , since  $V$  contains  $16 - 10 = 6$  triples. It is easy to verify that  $G$  is isomorphic to  $K_{33}$ .

If  $t$  has rank 3, then  $r = r_{ij}$ , where  $r_{ij}$  is defined in equation (3). The roots  $r_{ij}$ ,  $ij = 12, 23, 31$  when the equality holds in 4, form a triple  $v_0$  of rank 2. Obviously,  $v_0 \in V$ . We set  $V_0 = V - \{v_0\}$ . Since the root  $r_{ij}$  belongs to two triples  $v_0$  and  $(r_{ij}, r_i, r_j)$ , the remaining triples of  $V(r_{ij})$  belong to  $V_0$ . We saw that each root  $r \in R_0$ ,  $r \neq r_{ij}$  for  $ij = 12, 23, 31$ , belongs to two triples of  $R_0$ . Hence, in the case when  $t$  has rank 3, the graph  $G$  is well defined. It is easy to verify that  $G$  is isomorphic to  $(K_5 - 1)^*$  and  $v_0$  corresponds to a separating cut of cardinality 3.

Arcs of  $G$  are labeled naturally by roots from the set  $R_0$ . This labeling gives a representation of the cographic matroid of  $G$  by vectors of  $R_0$ .

Note that triples of rank 2 are equivalent under action of the automorphism group of  $P_V(D_4)$ . Similarly, all triples of rank 3 are equivalent under the automorphism group of  $P_V(D_4)$  extended by changing signs of roots. Hence, any explicit representations of the cographic matroids of the graphs  $K_{33}$  and  $(K_5 - 1)^*$  for fixed triples of rank 2 and 3 proves that above labeling give representations for all pairs of triples of rank 2 and 3.  $\square$

**6.4.  $L$ -types in  $\xi(\text{HYP}_5)$ .** From Proposition 4 we deduce that each  $L$ -domain corresponding to  $D_4$  is contained in  $64 = 4^3$  different  $L$ -types. 48 of them are of type II and 16 are of type III.

Recall that  $P_V(D_4)$  is free along lines spanned by roots of the root system  $D_{4,2}$ , vectors of which are parallel to diagonals of the cross-polytopes  $\beta_4$  of the Delaunay

partition of the lattice  $D_4$ . Note that bases related to the forms  $a(D_4) \in \xi(HYP_5)$  contain a diagonal of a  $\beta_4$ .

**Lemma 5.** *Let the basis related to  $a(D_4)$  contains a diagonal  $q \in D_{4,2}$  of a cross-polytope  $\beta_4$ . Then the  $L$ -domain of the parallelohedron  $P_q = P_V(D_4) + z(q)$  related to this basis, i.e. the  $L$ -domain of the form  $a(D_4) + \lambda f_q$ , does not belong to  $\xi(HYP_5)$ .*

*Proof.* The parallelohedron  $P_q$  has a non-zero width along the line  $l(q)$  parallel to  $q$ . The lattice  $L_q$  of  $P_q$  has a lamina  $H$  which is orthogonal to  $q$ . The lamina  $H$  separate the cross-polytope  $\beta_4$  with a diagonal  $q$  into two Delaunay polytopes, each being a pyramid with a base  $\beta_3$  orthogonal to  $q$  and lying in  $H$ . These two pyramids have the end-points of the diagonal  $q$  as apexes. Hence vertices of the basic simplex of  $a(D_4) + \lambda f_q$  belong to two distinct Delaunay polytopes. This implies that the  $L$ -domain of  $L_q$  does not belong to  $\xi(HYP_5)$ .  $\square$

**Proposition 5.** *The cone  $\xi(HYP_5)$  contains 12 principal  $L$ -domains, 120  $L$ -domains of type II and 40  $L$ -domains of type III, total 172  $L$ -domains.*

*Proof.* By Proposition 2,  $\xi(HYP_{4+1})$  contains  $\frac{1}{2}4! = 12$  principal  $L$ -domains.

The closure of an  $L$ -domain of types II and III is the convex hull of  $a(D_4)$  and a dicing facet  $F(U)$  related to a unimodular subset  $U \subset D_{4,2}$  of 9 vectors that are free for  $P_V(D_4)$ .

Each tile of the Delaunay tiling of the root lattice  $D_4$  is the regular four-dimensional cross polytope  $\beta_4$ . Any affine base of  $\beta_4$  contains exactly the two vertices  $w, w'$  of a diagonal and 3 vertices  $v_1, v_2, v_3$  chosen from the other diagonals. We have  $d_{\beta_4}(w, w') = 4$  and 2 for all other pairs. There are  $\binom{5}{2} = 10$  ways to choose a pair  $\{w, w'\}$  in a 5 elements set so there are exactly 10 rays  $a_i(D_4)$  representing  $D_4$  in  $HYP_5$ .

Every  $L$ -domain  $a_i(D_4)$  is contained in the closure of 64 primitive  $L$ -domains  $\mathcal{D}(U)$ , but not all of them are included in  $\xi(HYP_5)$ . Each  $L$ -domain  $\mathcal{D}(U)$  has the form  $\mathcal{D}(U) = \text{conv}(a_i(D_4) + F(U))$ , where the subset  $U \subset D_{4,2}$  is obtained by a deletion of a triple from  $D_{4,2}$ . By Lemma 5, the inclusion  $\mathcal{D}(U) \subset \xi(HYP_5)$  implies  $U$  does not contain the diagonal  $q$  which is a basic vector of  $a_i(D_4)$ . Let  $q = r_1 \in Q_1$ , then we have 4 choices for the root  $r_2$  in  $Q_2$ . For the third root either we have a rank 2 triple and  $r_3$  is completely determined or we have a rank 3 triple in which case there are 3 choices. This means that a ray  $a_i(D_4)$  is contained in 4  $L$ -domains of type II included in  $\xi(HYP_5)$  and 12  $L$ -domains of type III included in  $\xi(HYP_5)$ . This means that there are 120, respectively 40  $L$ -domains of type II, respectively III in  $\xi(HYP_5)$ .  $\square$

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